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Note

A Note on Stability Properties of Some Discrete-Time Systems

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The main objective of this note is to clarify some assumptions and results given in the paper entitled "Stability Properties of Positively Invariant Linear Discrete Time Systems" by Burgat and Benzaouia [*J. Math. Anal. Appl.* **143** (1989), 587–596].

1. INTRODUCTION

Burgat and Benzaouia [1] proposed the necessary and sufficient conditions for the geometrical characterization of the stability properties, such that asymptotic stability, stability, and instability [3], for a dynamical system described by

$$x_{k+1} = Ax_k; \qquad A \in \Re^{n \times n}, \tag{1}$$

for which matrix A was assumed to possess the property of leaving a proper cone $K \subset \Re^n$ positively invariant [2], [4], that is,

$$AK \subseteq K. \tag{2}$$

However, without an additional assumption regarding matrix A, the results proposed in [1] are incomplete. Indeed, these results only concern

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the class of the K-irreducible matrix A, but this hypothesis does not explicitly appear in the paper. Hence the objective of this note is to show that all results of [1] remain valid when matrix A is assumed to be K-irreducible.

This note is therefore organized as follows. Section 2 gives the definition of K-irreducibility and shows how this property can be characterized. Then Section 3 describes how this property behaves in different lemmas and theorems to make right the results of [1]. Finally, Section 4 draws conclusion and perspectives.

2. On the K-Irreducibility

Throughout this note, the notations of [1] are used.

To validate the results proposed in [1], the K-irreducibility assumed for matrix A of system (1) must be added. Section 3 shows how it works.

The following definitions are required.

Definition 2.1 [2]. Let K and $F \subseteq K$ be pointed closed cones. F is called a face of K if

$$x \in F, y \in K \text{ and } x - y \in K \Rightarrow y \in F.$$

Face F is nontrivial if $F \neq \{0\}$ and $F \neq K$.

Note that a pointed closed cone is defined by items (ii) and (vi) of Definition 2.3 in [1].

Definition 2.2 [2]. Matrix A satisfying property (2) is K-irreducible if and only if the only faces of K that it leaves invariant are $\{0\}$ or K-itself.

The main spectrum property of K-irreducibility can be characterized by the following theorem.

Theorem 2.1 [2]. *If A is K-irreducible then*

(i) *$\rho(A)$ is a simple eigenvalue and any other eigenvalue with the same modulus is also simple.*

(ii) *There is an eigenvector corresponding to $\rho(A)$ in* Int K_+, *and no other eigenvector (up to a scalar multiple) lies in K.*

The following notation is introduced (used in [2], [4]): a proper cone K induces a partial ordering in $\Re^n$ defined by $x \overset{K}{\geq} y$, which means that $x - y \in K$.

3. Stability Results

3.1. *Preliminary Results*

In a first stage, the preliminary results given in Section 3 of [1] are considered. All the lemmas (Lemmas 3.4 to 3.9) proven in this section remain valid and require no additional hypothesis of K-irreducibility for matrix A. Nevertheless, this new hypothesis allows us to complete the preliminary results of [1] in order to correct the main results. Especially some preliminaries are expressed in the cases when $\rho(A) = 1$ or $\rho(A) > 1$.

LEMMA 3.1. *If $\rho(A) = 1$ then $(C_+ \cap C_-)\backslash\{0\} \cap K_+ \neq \varnothing$.*

Proof. Let eigenvector x_0 of matrix A corresponding to $\rho(A) = 1$, x_0 is different from vector zero. From Lemma 3.3 of [1], $x_0 \in K_+$ and since $\rho(A) = 1$ we get $-\Pi x_0 = \Pi x_0 = 0$; therefore x_0 belongs to C_+ and C_-. Thus $(C_+ \cap C_-)\backslash\{0\} \cap K_+ \neq \varnothing$.

LEMMA 3.2. *If A is K-irreducible with $\rho(A) = 1$ then $C_+ = C_-$ and $(C_+ \cap C_-)\backslash\{0\} \cap \operatorname{Int} K_+ \neq \varnothing$.*

Proof. It readily follows from Theorem 2.1 and Lemma 3.1.

LEMMA 3.3. *If A is K-irreducible with $\rho(A) \geq 1$ then $\nexists \in \operatorname{Int} K_+$ such that $-\Pi x \neq 0 \in K_+$.*

Proof. Let x_0 be the eigenvector of matrix A corresponding to $\rho(A)$, which can be equal to 1 or >1; $x_0 \in \operatorname{Int} K_+$. Assume also that there exists a vector x belonging to $\operatorname{Int} K_+$ such that $-\Pi x \neq 0 \in K_+$. We can then construct the vector $y = \alpha x - x_0 \in \partial K_+$ with $\alpha > 0$; this scalar α always exists since x and x_0 belong to $\operatorname{Int} K_+$ (see Vandergraft [4]). From the assumptions, it follows that $0 \overset{K}{\leq} Ay = \alpha Ax - \rho(A)x_0 \overset{K}{\leq} \alpha x - x_0 = y$. In other words, vector y belongs to ∂K_+, $Ay \in \partial K_+$, contradicting the K-irreducibility assumption.

LEMMA 3.4. *If A is K-irreducible with $\rho(A) > 1$ then $\nexists x \in K_+\backslash\{0\}$ such that $-\Pi x \neq 0 \in K_+$.*

Proof. With respect to vectors $x \in \operatorname{Int} K_+$, the result follows from the above Lemma 3.3. Regarding vectors $x \in \partial K_+$, and since from Lemma 3.4 of [1] we get $-\Pi x \notin \operatorname{Int} K_+$ the only possibility to obtain $-\Pi x \in K_+$ is therefore to have $-\Pi x \in \partial K_+$, that is, $0 \overset{K}{\leq} Ax \overset{K}{\leq} x$. Vector x belonging to ∂K_+ contradicts the assumption of K-irreducibility of A.

LEMMA 3.5. *If $\rho(A) > 1$ then $(C_- \cap K_+)\backslash\{0\} \neq \varnothing$.*

Proof. Consider the eigenvector x_0 of matrix A corresponding to $\rho(A) > 1$. We have $x_0 \in K_+$ and thus $-\Pi x_0 = (1 - \rho(A))x_0 \in K_-$; therefore $x_0 \in C_-$, and Lemma 3.5 follows.

3.2. *Main Results*

Now consider the main results given in Section 4 in [1].

The first result proposed in [1, Theorem 4.1, p. 592] deals with the asymptotic stability of system (1)–(2). This result is not affected by the forgotten hypothesis. In other words, Theorem 4.1 of [1] is true whether or not matrix A is assumed to be K-irreducible. Consequently, all the results proposed in Section 5 of [1] do not need to be modified.

In the sequel it is shown that the K-irreducibility hypothesis for matrix A is needed to validate the results concerning the cases when $\rho(A) = 1$ and $\rho(A) > 1$.

LEMMA 3.6. *If* $(C_+ \cap C_-)\backslash\{0\} \cap K_+ \neq \varnothing$ *then* $\rho(A) \geq 1$.

Proof. The condition of this lemma means that there exists a vector $x \in K_+$ such that $-\Pi x = \Pi x = 0$, that is, $Ax = x$. Hence, this vector corresponds to an eigenvalue $\lambda_i(A) = 1$, but without additional conditions, it cannot be asserted that $\rho(A) = 1$, since there may exist another eigenvalue $\lambda_j(A)$, $|\lambda_j(A)| > 1$, that is, $\rho(A) > 1$.

Then the suitable theorem (corresponding to Theorem 4.2 in [1]) can be formulated as follows.

THEOREM 3.1. *The dynamical system* (1), *with* (2), *for which A is K-irreducible, is critically stable if and only if* $(C_+ \cap C_-)\backslash\{0\} \cap \operatorname{Int} K_+ \neq \varnothing$.

Proof. (If) The K-irreducibility hypothesis for matrix A allows us to conclude in the previous proof of Lemma 3.6 that $\rho(A) = 1$.

(Only if) It readily follows from Lemmas 3.1 and 3.2.

THEOREM 3.2. *If matrix A is K-irreducible with* $\rho(A) = 1$ *then the critical case is stable in the sense of LaSalle* [3].

Proof. The K-irreducibility of matrix A implies for $\rho(A) = 1$ to be a single eigenvalue from Theorem 2.1; further, from item (ii) of Lemma 3.3 in [1], we obtain the critical stability in the sense of LaSalle [3].

The above result proves the validity of Theorem 4.6 of [1], with the additional property of K-irreducibility; note that item (ii) of Theorem 4.6 of [1] becomes irrelevant because the K-irreducibility of matrix A implies $(C_+ \cap C_-)\backslash\{0\} \cap \operatorname{Int} K_+ = \varnothing$ when $\rho(A) = 1$.

In the sequel, we consider the case $\rho(A) > 1$.

LEMMA 3.7. *If* $(C_+ \cap K_+)\backslash\{0\} = \varnothing$ *then* $\rho(A) > 1$.

Proof. To assume that $(C_+ \cap K_+)\backslash\{0\} = \varnothing$ means that $\nexists x \in K_+$ such that $-\Pi x \in K_+$, or equivalently $\forall x \in K_+$, $-\Pi x \notin K_+$. Let x_0 be the eigenvector of A corresponding to $\rho(A)$, $x_0 \in K_+$, $x_0 \neq 0$. It follows that $-\Pi x_0 = (1 - \rho(A))x_0 \notin K_+$. Hence one must have $1 - \rho(A) \neq 0$ and $1 - \rho(A) \ngtr 0$, which implies $\rho(A) > 1$.

This lemma allows us to validate Theorem 4.4 of [1] when matrix A is K-irreducible as stated below.

THEOREM 3.3. *System* (1), *with* (2), *for which A is K-irreducible, is unstable* ($\rho(A) > 1$) *if and only if* $(C_+ \cap K_+)\backslash\{0\} = \varnothing$.

Proof. It readily follows from Lemmas 3.7 and 3.4.

4. CONCLUSION

In this note, the erroneous results detected in [1] have been corrected by adding the hypothesis of K-irreducibility for matrix A.

Then, it has been shown that the asymptotic stability result is independent of the K-irreducibility hypothesis. The other results on critical stability ($\rho(A) = 1$) and instability ($\rho(A) > 1$) have been validated by adding this hypothesis.

The generalization of the results of [1] to the case when matrix A is not assumed to be K-irreducible requires adding several preliminary results and developments. Consequently, this case cannot be treated in this note but will be proposed in a future paper.

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